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STAGES OF WITTGENSTEIN'S PHILOSOPHY OF MATHEMATICS

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Abstract

Wittgenstein's philosophy of Mathematics is concerned relatively few philosophers of Mathematics, historians of Philosophy from the XX century and analytically oriented philosophers. The paper deals with the periodization of Wittgenstein's philosophy of Mathematics. It is well known that Wittgenstein's philosophy of Mathematics does not correspond to the overall periodization of his work. We find the obvious context of Wittgenstein's grasping of value and Mathematics. Mathematics of Tractatus Logico - Philosophicus corresponds to the division of the sentences by meaning. Equations as the mathematical assertions are thus only apparent sentences (Scheinsätze) because they try to discuss the logical form. The claims of Mathematics are, in Wittgenstein's sense, absurd. According to the Tractatus, we have only one language, not a meta-language. Strict view of Mathematics is related to a rigorous view of values and ethics. Wittgenstein's philosophy of Mathematics works in the overall portfolio of opinions in many ways as anarchist. Wittgenstein's rejection of some mathematical objects is closely related to his understanding of the syllables: exuberance, meaningfulness. It is undoubtedly related to Wittgenstein's ethics.

Keywords: values, mathematics, Ludwig Wittgenstein, meaning, withdrawability

1. Introduction

"If we look at the major developments initiated by famous mathematicians at the end of the 19th and at the turn of the 20th century, such as, Cantor, Dedekind, Hilbert, Klein, Kronecker and Poincare, to mention but a few, we see that they engaged in what, by any reasonable standard, can be identified as philosophy of Mathematics." [1] Only few philosophers of Mathematics, historians of the Philosophy of the 20th century and analytical philosophers paid attention to Wittgenstein's philosophy of Mathematics. The first significant attempt was probably Kreisel's article dealing with Wittgenstein's opinion on the foundations of Mathematics [2] which was later followed by an article

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discussing Mathematics in the Tractatus [3]. The sociology of Mathematics as understood by Wittgenstein and Mannheim [4] and Wittgenstein's philosophy of Mathematics [5] were the topics of other studies published in the following decade. Monographs discussing the above topic could be counted on the fingers of one hand and the number of articles published in renowned scientific journals barely exceeds fifty articles. Some studies are strictly monothematic such as the study regarding Wittgenstein's reflection on the wood sellers scenario present in the 'Remarks of the Foundation of Mathematics' [6].

In the overall portfolio of opinions Wittgenstein's philosophy of Mathematics seems to be quite anarchistic [7]. Despite that some authors point out its usefulness from the viewpoint of the history of Mathematics, e.g. with respect to fuzzy logic [8] or the theory of Latin Squares used in Sudoku [9]. It is not our task to map Wittgenstein's opinions on philosophy of Mathematics. This has been already done [10-12]. It is clear that Wittgenstein cannot be associated with intuitionism, logicism or formalism. Some authors believe that "such an audacious rethinking in a paraconsistent framework may nowadays vindicate some of Wittgenstein's 'outrageous claims', which were dismissed too swiftly by commentators who dogmatically took the logic of Russell and Frege as the One True Logic" [13, p. 217]. As Steiner says "Wittgenstein's abiding view that mathematical propositions express rules has led to 'radical conventionalist' interpretations of his philosophy of Mathematics, according to which each mathematical theorem is a separate convention" [14, p. 3]. Our aim is more concrete. We want to try to identify the stages of Wittgenstein's philosophy of Mathematics.

2. Early Wittgenstein and Mathematics

It seems that before he wrote the Tractatus Wittgenstein had rejected Frege's opinion that numbers are objects. Before 1913 he had probably shared the same standpoint. The Tractarian philosophy of Mathematics has been fully mapped by Jakub Gomulka [15]. We are going to look more closely at some of its components. "Several authors have detected profound analogies between Kant and Wittgenstein. Their claims have been contradicted by scholars, such being the agreed penalty for attributions to authorities." [16] In terms of Kant's transcendental viewpoint Wittgenstein was influenced by Schopenhauer and Helmholtz. In the Tractatus Wittgenstein discusses some logical topics and metaphysics and he also presents his specific opinions on Mathematics. "Most of the remarks about Mathematics itself in the Tractatus are concentrated in the 6.2 – 6.24, but there are many related concepts, such as number, infinity, operation, and so on, that occur scattered through the length and breadth of the book." [3] It is interesting that he considers the theory of classes useless in the Tractarian philosophy of Mathematics. "This is connected with the fact that the generality which we need in Mathematics is not the accidental one." (6.031) [17] In the Tractatus he does not defend logicism. The basic mathematical term of the Tractatus is an equation. Wittgenstein believes that numbers are not the names of objects, they are their indexes. "What is important for our purposes about the account in the Tractatus is in any case not its exact details but the place it gave to arithmetical equations as attempts to encapsulate in symbolic form the tautologousness of various propositions." [18] Basically, equations are not predicable in a natural language. Wittgenstein says that this actually shows what is not predicable. It is possible to point at variables of a suitable type. It seems that Wittgenstein partially negatively reacts to Ramsey who tries to derive the basic term of class from logic. Wittgenstein raises objections to Russell's idea that Mathematics is based on logic and consists of tautologies. The views of the author of the Tractatus were quite different from the then standpoints of formal logic which were based on the works of Frege and Russell. He puts it clearly in 6.22 – logic postulates theses in tautologies and Mathematics in equations.

As Gomulka emphasises Wittgenstein was to a certain extent inspired by Frege, "the change from Verfahren Frege to Operation Wittgenstein is not merely a language change. The author of the Tractatus elaborated the thoughts of operations which contradicted the Frege – Russell's concept of function." [15, p. 79] Wittgenstein's criticism of Frege is based on the fact that logicism defines what is predicable [19]. Our philosopher provides a recursive definition of the sequence of natural numbers as exponents of operations on propositions. As Gomulka reminds us, with general numbers he applies a method which is used in connection with fields - what applies to one member of the range applies to the following member as well. In case Wittgenstein's concept of a number is based on the system where number is an exponent of operations, it is easy to perform a mathematical proof of simple mathematical operations. His theory of a number is a theory of ordinary numbers [7, p. 540]. From the viewpoint of Frege's criticism it is blending of variables, i.e. numbers and object variables which, according to Kolman [19], relate to the predefined field which eventually results in the Russell's paradox (or barber paradox in the popularised version). Kolman criticises Wittgenstein's Tractarian arithmetic because his arithmetic collapses in the finite world variant (describable by a system of elementary proposition with finite number of elements) [7, p. 543].

In Wittgenstein's understanding the assertions of Mathematics are actually absurd. According to the Tractatus there is just one language no metalanguage. Equations as assertions of Mathematics are only pseudo propositions (Scheinsätze) because they discuss logical form. Their status represents a problem as our philosopher considers them meaningless [15]. Gomulka further analyses the question of absurdity in Wittgenstein's Tractatus [15]. Wittgenstein considers propositions of logic and Mathematics meaningless because one cannot learn anything from them. Of course, they are not meaningless as propositions of classical Metaphysics. "In their case it is a legitimate and logically correct chain of signs, despite the fact that the last statement has no information value." [20, p. 133] The so-called self-destructive message of the Tractatus (6.54) claims that those who understand this Wittgenstein's book will consider it absurd. However, the message does not lead us to such radical conclusions as the new Wittgensteinians presented, i.e. that the whole Tractatus and its basic theses are intentionally self-destructive, for example [21]. This certainly is not true, moreover, as Trueman puts it "discussion of Tractarian logicism has wider implications for contemporary philosophy" [22, p. 309].

Wittgenstein deals with the famous Russell's paradox in the following way (Tractatus 3.333): function cannot be the argument because the pattern of the argument is already included in the sign, i.e. it cannot include itself in itself. "If, for example, we suppose that the function F(fx) could be its own argument, then there would be a proposition 'F(F(fx))', and in this the outer function F and the inner function F must have different meanings; for the inner has the form φ(fx), the outer the form ψ(φ(fx))." [17, p. 36] The solution of our philosopher rests in the argument according to which "inner and outer functions in proposition F(F(x)) must have different meanings as a result of which there is no paradox" [7, p. 539]. As Kolman reminds us, the operation does not have ontological character and it is not part of the proposition's meaning.

Gomulka claims that in case of Wittgenstein's philosophy of Mathematics one cannot speak of major changes from the development point of view, it is rather "continuation of basic intuition" [15, p. 91]. Beran assumes that periodization of the philosophy of Mathematics "does not fully overlap with periodization and breaking points of visible philosophy" [23].

3. Intermediate Wittgenstein and Mathematics

It was in the middle period when Wittgenstein devoted most of his attention to the philosophy of Mathematics. "One of the first thing – if not the first song – Wittgenstein wrote in philosophy following his exile in the wilderness of Lower-Austrian elementary schools is a criticism of Ramsey's paper The Foundation of Mathematics." [24] It is very significant that Wittgenstein renewed his interest in philosophy after hearing Brouwer's lecture. "Wittgenstein's remarks on contradictions in the foundation in Mathematics and in Logic are generally considered to be most obscure and paradoxical." [25] Middle Wittgenstein's philosophy of Mathematics is marked by constructivism. At that time Wittgenstein's standpoints were very rigorous and it can be said that he was a supporter of strong mathematical verificationism. Despite that Wittgenstein might thus be interpreting as criticizing a range of views in philosophy of Mathematics – for example, psychologism, logicism, Platonism, intuitionism, formalism, conventionalism – but not positively associated with any doctrine or position [26]. In Wittgenstein words Mathematics is done and created. It is nothing but syntax. As he says, Mathematics is calculus which performs operations. People construct concrete statements in accordance with the rules of the calculus. The mathematical constructivism is defended in a radical way. Mathematics does not need foundations. It is human practice. Despite that Wittgenstein's approach to Mathematics is more philosophical than sociological. "One can very well see that in 1929 Wittgenstein believed that arithmetic is more fundamental than logic." [27, p. 122] Wittgenstein mistrusted non-constructivist evidence. His anti-foundationalism is present both in the

Tractatus and in his middle period works. On the other hand, however, categorising Wittgenstein of any period represents a problem. As Ohtani says: "However, it can be argued that Wittgenstein is doing something very different, and that if his philosophy has something to offer to the contemporary scene, it does not consists in any 'ism' but in the way he clarifies our ways of conceiving of Mathematics" [28].

From the viewpoint of extensions and intensions the infinity represents a problem. As an extension, an aggregate is a set of elements. Infinite extension is not possible. Infinity is perceived via intension. Wittgenstein understands an infinite numerical series as an infinite possibility of finite numerical series. "Notation of an infinite class offers 'a recursive rule' for inductive development of the finite basis." [23, p. 219] Wittgenstein thus rejects the notion of actual infinity. Intension and extension cannot be merged or substituted. We cannot speak of infinite conjunctions and disjunctions either. Infinite extension is only potentially infinite and it is not present in God's mind either. Similarly, existential quantificator cannot be related to an infinite number of entities. "Wittgenstein is, we are told, a strict finitist, who holds that the only comprehensible and valid kind of proof in mathematics takes the form of intuitively clear manipulations of concrete object." [29, p. 50] He also criticises the assertion that some infinite sets are greater in cardinality than other infinite sets. "According to Wittgenstein's analysis the Euclidian proof of the infinity of primes suffers from a confusion of proof and prose." [30, p. 75]

Wittgenstein did not sympathise with intuitionism due to the law of excluded middle. There are opinions according to which some of Wittgenstein's statements are close to intuitionism [31], but "there are relevant differences between intuitionist mathematics and Wittgenstein's philosophy of Mathematics in the cases of the Law of Excluded Middle, proofs and the foundation of Mathematics" [32, p. 168]. Brouwer speaks of the existence of mathematically undecided propositions. For Wittgenstein they are non-sense. The way how to confirm the validity of a mathematical proposition is to prove it. Mathematical propositions are constructs and as such they are distinguishable. With respect to the above mentioned Wittgenstein considers proving of the Fermat's great theorem senseless. He acknowledges mathematical induction. However, his understanding of induction is very specific. For him it is a way to show the other possible steps in the sequence. Mathematical induction thus only pretends to be a rule. In the strict sense of the word it is not a mathematical rule at all.

Wittgenstein has a specific approach to irrational numbers. In his opinion, purely irrational numbers cannot be viewed extensionally. Even if the number is extensional by its entry, its meaning must be intentional. For him, irrational numbers are material for filling in spaces to be used by mathematicians studying the continuum. Criterion for determining the meaning of irrational numbers is their comparison with rational numbers and determining which of them is bigger. In his opinion, unlimited decimal development is a dead rule of a live rule which expresses a number. Numbers which are necessary only for filling in the continuum Wittgenstein calls false irrational numbers. Irrational numbers themselves have meaning only as rules. “The extensional set - theoretic conception of a real number does not give a foundation for real analysis either.” [33] Infinite extensions must be an insurmountable obstacle for algorithm. On the other hand, our philosopher accepts complex numbers because they are constructed and they are related to practice. Irrational numbers, however, are just a dangerous pseudo concept for him.

Wittgenstein is very critical to mathematical propositions as well. “Wittgenstein, in his middle period, clearly maintains, that the sense of a mathematical proposition is determined by its method of verification. As a result, if a mathematical proposition has not been proven (i. e., if it is as yet undecided), then it is meaningless (and so, strictly speaking, it is not a proposition). Thus this strong verificationist view of mathematical proposition implies that there are no conjectures in Mathematics.” [34, p. 412] Unlike Turing, Wittgenstein rejects any thoughts regarding analogy of learning in Mathematics and Physics. He identifies the meaning of a mathematical proposition with the method of verification. There is no space for hypotheses in Mathematics because there are no rules for defining the answers. As a result, there is no way how to search for the answer. The only proof Wittgenstein accepts is the constructivist proof.

Wittgenstein criticises the theory of aggregates both from the viewpoints of constructivism and finitism. “Usually, a set theory is developed in the framework of the Zermelo – Fraenkel axiom system, including the axiom of choice.” [35] Wittgenstein directly attacks the foundations of the set theory. “Set theory, for Wittgenstein, is not a theory about sets, not if sets are conceived as existing prior to the theory.” [36] Aggregates cannot be limitless. It is not possible to think of an aggregate of all numbers of any kind. From philosophical viewpoint he considers the theory of aggregates shallow. He uses a similar basis to criticise Dedekind’s definition of infinite class. Beran concludes that Wittgenstein’s constructivism is normative [23, p. 237]. In his opinion the middle and late periods of Wittgenstein in Philosophy merge.

In his middle period Wittgenstein writes also about mathematical induction and describes it as shallow and inconsistent. This problem is related to the above-mentioned criticism of mathematical propositions. “What ‘mathematical questions’ share with genuine questions, is simply that they can be answered.” (§ 151) [37] This, however, is not true in case of mathematical induction. Correspondence including a proof by means of induction represents an ungrounded induction step. A general answer cannot be supported by inductive evidence. Wittgenstein accepts a proof which can be found in the entire system of calculations.

“Clearly Wittgenstein regards at least some of pure mathematics as quite easy to relate to applications in empirical propositions.” [38, p. 4145] Mathematics is studied on the basis of language analysis. Mathematicians use concepts like number, proof, order, etc. which can be found in ordinary language as well. The grammar of mathematical propositions can be explained philosophically on the basis of language analysis. In his middle period, Wittgenstein's philosophy of Mathematics has its source in the philosophy of language. The philosophy of language itself thus becomes the basis for a philosophy of Mathematics.

4. Late Wittgenstein and Mathematics

Let us focus on late Wittgenstein now. Late Wittgenstein considers language a medium as well as provider of topics for thinking [39]. Miguel thinks [40] that Wittgenstein´s philosophy of Mathematics of the late period is securely placed in an ivory tower. In On Certainty "there are important parallels between (some of) Wittgenstein's views on mathematical sentences and central aspects of his account of certainties" [41, p. 141]. As Schlegel points out, according to late Wittgenstein language is human practice which abides by the rules. It uses constructive rules and calculus just like Mathematics [42].

It is generally known that the late Wittgenstein period is marked by criticism of Gödel's theorems. Wittgenstein's philosophy of Mathematics requires interpretation of logical sentences. When we ask: 'in which system can this be proven?', we must also ask: 'in which system is this true?'. Wittgenstein believes that without this question the entire proof is just syntactic, however, it does not say whether the sentence is or is not true on the basis of semantics. Kurt Gödel himself admits that a problem occurs in case the symbols in his formal proof are replaced by entities with meanings. "If we want the Gödel's theorem to express something, we must accept at least that interpretation where π tells the truth of some chains, and thus interprets some of the chains of the initial system as true or untrue (and not only as provable or not provable)." [43] For Wittgenstein it is essential to ask in which system the sentences are true. There is a problem, however. With Gödel's proof it is not necessary to assume that the theorems are true. If we interpret the said theorems, we cannot automatically ensure that their truthfulness in our interpretation will be truthfulness in a sense given to them by someone else by interpreting the original chain of the system. The character of interpretation of syntactically given theorems will determine whether they are true or not. From this meta-mathematical viewpoint Wittgenstein draws a conclusion that Gödel's theorems are a nice proof, nevertheless, they actually say nothing about their truthfulness. Due to the above mentioned he labels the entire process of proving the theorems' trueness insufficient. In Wittgenstein's words it is an attempt to speak the unspeakable and thus the entire proof is basically meaningless and it is not what Kurt Gödel thought it was. He insisted on the fact that in case of Gödel's theory one cannot speak of formal language. Wittgenstein claims that evidence of syntactic nature cannot have any non-mathematical meaning. And his criticism is even deeper. He thinks that Gödel's proof is meaningless. However, he does not contradict it by Hilbert's deductive completeness. It is not possible to speak of the meaningless when we speak of the true and untrue. The system of natural language is completely unspeakable and incomplete for Wittgenstein. Values cannot be expressed meaningfully in the natural language therefore one should not talk about certain matters. Gödel demonstrated that our formal system cannot use everything mathematical reality includes. Similarly, early Wittgenstein claims that our linguistic systems cannot use everything non-mathematical reality includes [44].

In Philosophical Investigation late Wittgenstein continues to criticise logicism, however, he looks at it from a different viewpoint. His criticism is connected with the problem of following the rules. He demonstrates the problem on the sequence of numbers in which each number increases by 2, for example 0, 2, 4, 6, 8, etc. Even small children can understand such sequence. However, it is not possible to imagine an infinite number of additions. A problem might occur in case the rules change unexpectedly (e.g. specific state of brain). In such case it would not be possible to proceed pursuant to the true/false criterion but only on the basis of inclination of an individual. At this point it is necessary to explain the terms rule and regularity.

According to Brandom [45] the term regularity is implicitly present in the sequence itself. In such case the sequence can develop in various ways which is a problem. Such standpoint is called regularism. Other standpoint - regulism says that the rule exists explicitly and if sequence is different, it is simply in contradiction with the explicit rule. In Wittgenstein's opinion, this can again lead to a problem. Comparison of a rule and sequence of numbers, which differs from it, needs to be done by means of another rule which can lead to regress. It reminds us of Plato's Third Man Argument. If all rules were explicit, as it is in logicism, there would be a threat of infinite regress. It is thus impossible to verbalise everything. It is a very strong criticism of logicism although Wittgenstein looks at the matter from a different angle.

Late Wittgenstein believes mathematics was invented by a man "the mathematician is not a discoverer: he is an inventor" (RFM, Appendix II, §2) [46]. He continues to reject Platonism in Mathematics. He basically keeps his constructivist position. Similarly to his middle period he considers Mathematics non-referential and syntactical. Wittgenstein does not want to regard mathematical propositions as mathematical objects. If we regard Mathematics as exploration of subjects, it is already alchemy (RFM § 16) [46, p. 153].

Wittgenstein's late period did not witness any changes regarding his rejection of finitism or irrational numbers either. "What harm is done e.g. by saying that God knows all irrational numbers? Or: that they are already there, even though we only know certain of them? Why are these pictures not harmless? For one thing, they hide certain problems." (RFM VII, § 41) [46, p. 42] Neither God's mind can bring finitism towards actual infinity. (For example, compare with Kierkegaard's interpretations of God and finality [47-54]. For an older interpretation of God's relationship to creation (and His self-limitation) see [55, 56].)

Mathematical assertion considers mathematical proposition meaningful in case it is part of a calculus, i.e. if there exists a procedure for decidability. Algorithmic decidability remains the basic criterion. Only sense in which an undecided mathematical proposition can be decidable is in the sense that we know how to decide it by means of an applicable decision procedure (RFM VII, § 40) [46, p. 41]. "According to Wittgenstein, mathematical language has a normative role in the application of mathematics to systems of non-mathematical objects in that mathematical sentences may function as 'norms of descriptions', deciding which claims on the non-mathematical objects under consideration are meaningful and which not." [57, p. 13]

In his late period Wittgenstein criticizes the theory of aggregates and he adds that if one of the purposes of the theory of aggregates is laying the foundations of Mathematics then Mathematics does not need any foundation at all. Paradoxically, just like Poincaré he rejects diagonal proof because diagonal is explained as a proof of greater and lesser infinite sets. Wittgenstein calls this proof a hocus-pocus. We cannot take into consideration numbers which are bigger than the infinite. Cantor´s diagonal thus is not a proof of non-denumerability. Cantor has shown that we can construct "infinitely many" diverse systems of irrational numbers, but we cannot construct an exhaustive system of all the irrational numbers (RFM II, § 29).

Wittgenstein´s later periods witnessed some minor changes. "Wittgenstein introduces a new term 'language games' where Mathematics will be just one of the language games which are parts of our forms of life so different from the set, construct and transfer of mathematical knowledge of empirical sciences." [58, p. 82] Wittgenstein comes with simple sign – games and mathematical language games. The influence of the Remarks on Colour and Philosophical Investigation can be clearly felt in this context. The features of Mathematics should also be used outside Mathematics for the mathematical language games to make sense. It is something like an echo of the final sentences of the Tractatus. In this sense the extra – mathematical application is conditio sine qua non. Wittgenstein wants to have and use many different forms of language games. He considers his own theory of aggregates just a formal sign – game. Wittgenstein's opinion that mathematics is a matter of syntax prevailed also in his late period.

5. How much is the stages of Wittgenstein's philosophy of Mathematics? - attempt to conclude

Let us think about periodization of Wittgenstein's philosophy of Mathematics. Some of his philosophical standpoints never changed. Steiner points out [59] that he never abandoned the concept of objectivity of logic. Other commentators, for example Hacking "suggest an alternative to a traditional philosophical conceptualisation of chat mathematics 'is', and chat is about" [60].

It is quite easy to see the distinctions between early and middle Wittgenstein in the philosophy of Mathematics. Tractarian philosophy of Mathematics is specific. The basic mathematical terms are equation and equality. Wittgenstein did not defend logicism. In his opinion equations cannot be expressed in natural language and they are only pseudo-propositions. We need the propositions of Mathematics to derive other propositions from them. Wittgenstein defines natural numbers as exponents of operations on propositions. Wittgenstein's philosophy of Mathematics of the early period is an integral part of the Tractatus. Despite several common features Tractarian philosophy of Mathematics shared with late philosophy of Mathematics (criticism of logicism, applicability of Mathematics outside the sphere of pure Mathematics, etc.). The Tractarian philosophy of Mathematics differs from destructive and critical features of the said philosophy in later periods.

One can divide Wittgenstein's general philosophy into three periods. It is not possible, however, in the area of philosophy of Mathematics. We agree with Beran [23] that it is not basically possible to distinguish between the middle and late periods of Wittgenstein's philosophy of Mathematics. In the late period one can observe the occurrence of new topics (certainty in Mathematics, mathematics as a language game, etc.). Nevertheless, Wittgenstein keeps his hypercriticism. He continues to refuse the terms of actual infinity, irrational numbers or mathematical induction. He criticises the theory of aggregates as well as Gödel's theorems. Wittgenstein did not change anything about the radicalism of his middle period. In his opinion, Mathematics is a matter of syntax. In his middle period he speaks of its constructing, in his late period he says that Mathematics is a man's invention and keeps his anti-Platonism. The occurrence of new topics was not a significant change. There were some minor problems which late Wittgenstein evaluated differently than middle Wittgenstein. However, these opinions had no impact on his overall understanding of Mathematics and its content. The most significant change was the reintroduction of a criterion outside Mathematics for differentiating between mathematical language games and sign – games. This criterion was present in his first period but it is absent in his middle period. Wittgenstein's restrictive, rigorous and basically constructivist view did not change since the turn of the 1920s and 30s. Therefore we dare to say that from the viewpoint of philosophy of Mathematics there are only two stages. In general, the first stage corresponds with the period of the Tractatus and its preparation. The second stage started after Wittgenstein had renewed his activities in the field of Philosophy following his teaching at Austrian elementary schools and in terms of philosophy of Mathematics this stage lasted until his death. In general, we agree with dividing Wittgenstein's work into early, middle and late periods, nevertheless, with respect to philosophy of Mathematics the middle and late periods of his general philosophical work melt into one.

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